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2009 J. Phys. A: Math. Theor. 42 454021

(http://iopscience.iop.org/1751-8121/42/45/454021)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 454021 (15pp) doi:10.1088/1751-8113/42/45/454021

On some class of reductions for the Itoh–Narita–Bogoyavlenskii lattice

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Received 26 February 2009, in final form 2 June 2009
Published 27 October 2009
Online at stacks.iop.org/JPhysA/42/454021

Abstract
We show a broad class of constraints compatible with the Itoh–Narita–Bogoyavlenskii lattice hierarchy. All these constraints can be written in the form of a discrete conservation law $I_{i+1} = I_i$ with an appropriate homogeneous polynomial discrete function $I = I[a]$.

PACS number: 02.30.Ik

1. Introduction

The aim of this paper is to show explicitly some class of constraints compatible with the extended Volterra lattice

$$a_i' = a_i\left(\sum_{j=1}^{n} a_{i-j} - \sum_{j=1}^{n} a_{i+j}\right), \tag{1}$$

which we consider as a single evolution equation on the unknown function $a_i \equiv a(i, x)$ of discrete variable $i \in \mathbf{Z}$ and continuous variable $x \in \mathbf{R}$. For any $n \geqslant 1$, this equation is known to be integrable discretization for the Korteweg-de Vries equation [5]. Narita in the work [9], making use of Hirota's method, showed that the extended Volterra lattice admits soliton solutions. In [6] Itoh considered Lotka–Volterra systems which are equivalent to equation (1) supplemented by the specific periodicity condition $a_{i+2n+1} = a_i$. In what follows we call equation (1) the Itoh–Narita–Bogoyavlenskii (INB) lattice. Perhaps the most interesting case from the point of applications is $n = 1$, corresponding to the Volterra lattice [7, 17].

Equation (1) is known to admit the hierarchy of pair-wise commuting generalized symmetries which, as is shown in the paper, can be written in the form

$$\partial_s a_i = (-1)^s a_i\left(S_{sn-1}^{(n,s)}(i-(s-1)n+1) - S_{sn-1}^{(n,s)}(i-sn)\right), \tag{2}$$

where ∂_s stands for derivative with respect to the evolution parameter t_s with $s \geqslant 2$. The functions $S_s^{(n,l)}[a]$ will be explicitly defined in section 3. The INB equation itself can be written in the form (2) with $t_1 = x$ and $S^{(n,1)}_{n-1} = \sum_{j=1}^{n} a_{i+j-1}$. When the hierarchy is represented as (2), it is clear that the stationarity condition attached to some evolution parameter t_s, can be written as the periodicity condition $S^{(n,s)}_{sn-1}(i+n+1) = S^{(n,s)}_{sn-1}(i)$. It turns out that there exists a wide class of homogeneous polynomial discrete functions $I = I[a]$ defined by[1]

$$I_i = \sum_{(j_1,\ldots,j_{l+1})\in J} a_{i+j_1}\cdots a_{i+j_{l+1}}$$

for which the periodicity condition $I_{i+T} = I_i$, with respective period $T \in \mathbf{Z}$ is consistent with the INB lattice and its hierarchy. The set of all invariant constraints considered in the paper naturally includes periodicity conditions $a_{i+T} = a_i$.

We consider equation (1) and its hierarchy as a simplest case of reduction of the so-called Darboux-KP (DKP) chain hierarchy which in fact is a bi-infinite sequence of KP hierarchies and our main goal, in fact, is to approve our previous results [15] on this simple example. For completeness, we give, in the next section, preliminaries on our approach to investigate some class of integrable lattices related to the KP hierarchy. In section 3, we show compatible constraints for equation (1) in its explicit form in theorem 3. Section 4 is devoted to the Volterra lattice and its reductions. We write, in this section, attached systems of ordinary differential equations generated by corresponding constraints and its discrete symmetry transformations. Also we claim the relations defining spectral curves associated with the Lax matrices.

2. Preliminaries on the DKP chain hierarchy and its invariant submanifolds

2.1. DKP chain hierarchy

In [13, 14] we have developed an approach in which a broad community of integrable differential-difference equations (lattices) are related to the KP hierarchy. In a paper [15] we have shown that these integrable lattices admit a wide class of constraints compatible with all higher flows of its hierarchy. An objective of this section is to provide the reader by information about the DKP chain hierarchy and its reductions.

Integrable lattices in our geometric set-up naturally appear as a result of reductions of a bi-infinite sequence of KP hierarchies whose equations of motion we write in the form of two evolution generating equations [8]

$$\partial_s h(i) = \partial H^{(s)}(i), \tag{3}$$

$$\partial_s a(i) = a(i)(H^{(s)}(i+1) - H^{(s)}(i)). \tag{4}$$

The first relation (3) yields evolution equations of the KP hierarchy in the form of local conservation laws [18]. Laurent series: generating functions for conserved densities and the corresponding fluxes of the KP hierarchy

$$h(i) = z + \sum_{k\geqslant 2} h_k(i) z^{-k+1} \qquad \text{and} \qquad H^{(s)}(i) = z^s + \sum_{k\geqslant 1} H^s_k(i) z^{-k},$$

are related with KP wavefunctions

$$\psi_i = \left(1 + \sum_{k\geqslant 1} w_k(i) z^{-k}\right) \exp\left(\sum_{s\geqslant 1} t_s z^s\right)$$

as

$$h(i) = \partial\psi_i \cdot \psi_i^{-1} \qquad \text{and} \qquad H^{(s)}(i) = \partial_s \psi_i \cdot \psi_i^{-1},$$

[1] Here $J \subset \mathbf{Z}^{l+1}$ is some finite indexing set.

respectively. In turn, the Laurent series $a(i) = z + \sum_{k\geqslant 1} a_k(i) z^{-k+1}$ is calculated as $a(i) = z\psi_{i+1} \cdot \psi_i^{-1}$. We call equations (3) and (4) the DKP chain hierarchy. It is useful to rewrite generating equation (4) in the form of the differential-difference conservation law

$$\partial_s \xi(i) = H^{(s)}(i+1) - H^{(s)}(i)$$

with

$$\xi(i) = \ln a(i) = \ln z + \sum_{k\geqslant 1} a_k(i) z^{-k} - \frac{1}{2}\left(\sum_{k\geqslant 1} a_k(i) z^{-k}\right)^2 + \frac{1}{3}\left(\sum_{k\geqslant 1} a_k(i) z^{-k}\right)^3 - \cdots$$
$$\equiv \ln z + \sum_{k\geqslant 1} \xi_k(i) z^{-k}.$$

Thus, more exactly, equations (3) and (4) can be written as follows:

$$\partial_s h_k(i) = \partial H^s_{k-1}(i), \qquad \partial_s \xi_k(i) = H^s_k(i+1) - H^s_k(i).$$

To establish relationship of integrable lattices such as INB lattice (1), Shabat dressing lattice [10], Toda lattice [16], Belov–Chaltikian lattice [3] and so on, with the KP hierarchy, the following two theorems are useful.

Theorem 1 [13]. *The submanifold $\mathcal{S}^n_{l-1}$ defined by the condition*

$$z^{l-n} a^{[n]}(i) \in \mathcal{H}_+(i), \qquad \forall\, i \in \mathbf{Z} \tag{5}$$

is tangent with respect to the DKP chain flows defined by (3) and (4).

Theorem 2 [14]. *The chain of inclusions of invariant submanifolds*

$$\mathcal{S}^n_{l-1} \subset \mathcal{S}^{2n}_{2l-1} \subset \mathcal{S}^{3n}_{3l-1} \subset \cdots \subset \mathcal{S}^{kn}_{kl-1} \subset \cdots$$

is valid.

Here, by definition

$$a^{[s]}(i) = \begin{cases} \prod_{j=1}^{s} a(i+j-1), & s \geqslant 1 \\ 1, & s = 0 \\ \prod_{j=1}^{|s|} a^{-1}(i-j), & s \leqslant -1 \end{cases}$$

are discrete Faà di Bruno iterates of Laurent series $a(i)$. It is obvious that the coefficients $a^{[s]}_j$ defined through the relation[2]

$$a^{[s]} = z^s + \sum_{j\geqslant 1} a^{[s]}_j z^{s-j}$$

are the discrete functions of $(a_1, \ldots, a_j)$. These functions are related with each other by the obvious relation

$$a^{[s_1+s_2]}_k(i) = a^{[s_1]}_k(i) + \sum_{j=1}^{k-1} a^{[s_1]}_j(i) a^{[s_2]}_{k-j}(i+s_1) + a^{[s_2]}_k(i+s_1) = (s_1 \leftrightarrow s_2). \tag{6}$$

Here the symbol $(s_1 \leftrightarrow s_2)$ denotes the same right-hand side of this relation but with mutually replaced s_1 and s_2. This relation will be extensively used throughout the paper.

We observe that the condition (5) can be written in the form of the following generating relation:

$$z^{l-n} a^{[n]} = H^{(l)} + \sum_{k=1}^{l} a^{[n]}_k H^{(l-k)}.$$

[2] Here and in what follows we use simplified notations $a^{[s]} \equiv a^{[s]}(i)$, $a^{[s]}_j \equiv a^{[s]}_j(i)$ in formulae which contain no shifts with respect to discrete variable $i \in \mathbf{Z}$.

2.2. nth discrete KP hierarchy and its reductions

When restricting the DKP chain hierarchy on $\mathcal{S}_0^n$, all the coefficients H_k^s become discrete polynomial functions of $(a_1, \ldots, a_{k+s})$ defined by [13, 15]

$$H_k^s = F_k^{(n,s)} \equiv a_{k+s}^{[sn]} + \sum_{j=1}^{s-1} q_j^{(n,sn)} a_{k+s-j}^{[(s-j)n]}, \tag{7}$$

where $q_k^{(n,r)} = q_k^{(n,r)}[a_1, a_2, \ldots, a_k]$, by definition, are polynomial discrete functions defined through the generating relation

$$z^r = a^{[r]} + \sum_{j\geqslant 1} q_j^{(n,r)} z^{j(n-1)} a^{[r-jn]} \tag{8}$$

which yields

$$a_k^{[r]} + \sum_{j=1}^{k-1} a_{k-j}^{[r-jn]} q_j^{(n,r)} + q_k^{(n,r)} = 0, \qquad \forall\, k \geqslant 1. \tag{9}$$

Let us write the first few $q_k^{(n,r)}$

$$q_1^{(n,r)} = -a_1^{[r]}, \qquad q_2^{(n,r)} = -a_2^{[r]} + a_1^{[r]} a_1^{[r-n]},$$

$$q_3^{(n,r)} = -a_3^{[r]} + a_1^{[r]} a_2^{[r-n]} + a_1^{[r-2n]} a_2^{[r]} - a_1^{[r]} a_1^{[r-n]} a_1^{[r-2n]}.$$

It can be checked that a more general relation than (9), namely [15]

$$a_k^{[r]}(i) + \sum_{j=1}^{k-1} a_{k-j}^{[r-jn]}(i) q_j^{(n,r-p)}(i+p) + q_k^{(n,r-p)}(i+p) = a_k^{[p]}(i) \tag{10}$$

with any $p \in \mathbf{Z}$ is valid. Solving this in favor of $q_k^{(n,r-p)}(i+p)$ yields

$$a_k^{[p]}(i) + \sum_{j=1}^{k-1} q_j^{(n,r-(k-j)n)}(i) a_{k-j}^{[p]}(i) + q_k^{(n,r)}(i) = q_k^{(n,r-p)}(i+p).$$

One sees that, when restricting on $\mathcal{S}_0^n$, the DKP chain hierarchy is reduced to evolution equations in the form of the differential-difference conservation law

$$\partial_s \xi_k(i) = F_k^{(n,s)}(i+1) - F_k^{(n,s)}(i), \tag{11}$$

where $F_k^{(n,s)}$ is given by (7). We refer to these equations with some fixed $n \geqslant 1$ as the nth discrete KP hierarchy. It is worth noting that these equations, in fact, appear as a result of the restriction of the DKP chain hierarchy on $\mathcal{S}_0^n$ and the subsequent projection of dynamics on the space $\mathcal{M}$ whose points are defined by infinite number of functions of discrete variable $(a_1, a_2, \ldots)$. One can say that $\mathcal{M}$ has an infinite functional dimension. Let us denote by $\mathcal{M}_k$ the space whose points are defined by the finite number $(a_1, \ldots, a_k)$ of functions of discrete variable i and $\pi_k : \mathcal{M} \mapsto \mathcal{M}_k$ being a natural projection. It is obvious that the reduction of the DKP chain hierarchy on the intersection $\mathcal{S}_0^n \cap \mathcal{S}_{l-1}^p$ is equivalent to restriction of the flows given by (11) on some submanifold $\mathcal{M}_{n,p,l} \subset \mathcal{M}$. This submanifold is defined by infinite number of algebraic equations [15]

$$J_k^{(n,p,l)}[a_1, \ldots, a_{k+l}] = 0, \qquad k \geqslant 1 \tag{12}$$

with

$$J_k^{(n,p,l)}(i) = a_{k+l}^{[p]}(i) - a_{k+l}^{[ln]}(i) - \sum_{j=1}^{l-1} q_j^{(n,ln-p)}(i+p) a_{k+l-j}^{[(l-j)n]}(i).$$

Observe that in the case $p = ln$ the relations $J_k^{(n,ln,l)} = 0$ are identities and therefore produce no nontrivial submanifold of $\mathcal{M}$. This is so because $\mathcal{S}_0^n \subset \mathcal{S}_{l-1}^{ln}$ thanks to theorem 2.

Taking into account (10) we can also write

$$J_k^{(n,p,l)}(i) = q_{k+l}^{(n,ln-p)}(i+p) + \sum_{j=1}^{k-1} a_j^{[-(k-j)n]}(i) q_{k+l-j}^{(n,ln-p)}(i+p). \tag{13}$$

Let us denote $Q_k^{(n,p,l)}(i) = q_{k+l}^{(n,ln-p)}(i+p)$ and therefore one can rewrite the relation (13) as

$$J_k^{(n,p,l)} = Q_k^{(n,p,l)} + \sum_{j=1}^{k-1} a_j^{[-(k-j)n]} Q_{k-j}^{(n,p,l)}. \tag{14}$$

It is evident that the submanifold $\mathcal{M}_{n,p,l}$ can be equivalently defined by algebraic equations $Q_k^{(n,p,l)}[a_1, \ldots, a_{k+l}] = 0$. Solving (14) in favor of $Q_k^{(n,p,l)}$ yields

$$Q_k^{(n,p,l)} = J_k^{(n,p,l)} + \sum_{j=1}^{k-1} q_j^{(n,-(k-j)n)} J_{k-j}^{(n,p,l)}. \tag{15}$$

By definition, one has $Q_k^{(n,p+n,l+1)}(i) = Q_{k+1}^{(n,p,l)}(i+n)$. Making use of this relation, (10) and (15), one can prove that

$$J_k^{(n,p+n,l+1)}(i) = J_{k+1}^{(n,p,l)}(i+n) + \sum_{j=1}^{k} a_j^{[n]}(i) J_{k-j+1}^{(n,p,l)}(i+n). \tag{16}$$

For $Q_k^{(n,p,l)}$ we are able to write the following evolution equations

$$\begin{aligned} D_{t_s} Q_k^{(n,p,l)}(i) &= Q_{k+s}^{(n,p,l)}(i+sn) + \sum_{j=1}^{s} q_j^{(n,sn)}(i+p) Q_{k+s-j}^{(n,p,l)}(i+(s-j)n) \\ &\quad - Q_{k+s}^{(n,p,l)}(i) - \sum_{j=1}^{s} q_j^{(n,sn)}(i-(s+k-j)n) Q_{k+s-j}^{(n,p,l)}(i) \end{aligned}$$

which is easily derived from

$$\begin{aligned} D_{t_s} q_k^{(n,r)}(i) &= q_{k+s}^{(n,r)}(i+sn) + \sum_{j=1}^{s} q_j^{(n,sn)}(i) q_{k+s-j}^{(n,r)}(i+(s-j)n) \\ &\quad - q_{k+s}^{(n,r)}(i) - \sum_{j=1}^{s} q_j^{(n,sn)}(i+r-(k+s-j)n) q_{k+s-j}^{(n,r)}(i). \end{aligned} \tag{17}$$

The latter, in turn, comes from the Lax equation attached to the linear problem for the wavefunction $\Psi = \{\psi_i : i \in \mathbf{Z}\}$ coded in (8) [12].

As was shown in [15], there exist weaker than (12) conditions invariant with respect to nth discrete KP hierarchy equations (11). Desired constraints are written as periodicity conditions

$$I_k^{(n,p,l)}(i+n) = I_k^{(n,p,l)}(i) \tag{18}$$

with

$$\begin{aligned} I_k^{(n,p,l)}(i) &= J_k^{(n,p,l)}(i) + \sum_{j=1}^{k-1} a_j^{[-p]}(i+p) J_{k-j}^{(n,p,l)}(i) \\ &= Q_k^{(n,p,l)}(i) + \sum_{j=1}^{k-1} a_j^{[-p-(k-j)n)]}(i+p) Q_{k-j}^{(n,p,l)}(i). \end{aligned}$$

Moreover, if the conditions (18) are valid then $I_k^{(n,p,l)}$ do not depend on evolution parameters t_s, i.e., $D_{t_s} I_k^{(n,p,l)} \equiv 0, \forall s \geqslant 1$. For these functions, we can write

$$I_k^{(n,p+n,l+1)}(i) = I_{k+1}^{(n,p,l)}(i+n) - a_k^{[-p-n]}(i+p+n) I_1^{(n,p,l)}(i+n)$$

in parallel with (16). It can be shown that the relationship of $I_k^{(n,p,l)}$'s with wave KP functions is given by

$$\sum_{j\geqslant 1} I_j^{(n,p,l)}(i) z^{-j} = z^l - \frac{1}{\psi_{i+p}} \left(z^l \psi_{i+ln} + \sum_{j=1}^{l} z^{l-j} q_j^{(n,ln-p)}(i+p) \psi_{i+(l-j)n} \right). \tag{19}$$

If d is any divisor of n, then the set of conditions $I_k^{(n,p,l)}(i+d) = I_k^{(n,p,l)}(i)$ also produces an invariant submanifold of the nth discrete KP hierarchy. We denote the corresponding submanifold by $\mathcal{N}_{n,p,l}^d$. It is evident that $\mathcal{M}_{n,p,l} \subset \mathcal{N}_{n,p,l}^d \subset \mathcal{N}_{n,p,l}^n$.

In the following section we consider invariant constraints for the Narita–Bogoyavlenskii lattice which corresponds to the restriction of the nth discrete KP hierarchy on $\mathcal{M}_{n,n+1,1}$. Our aim to write invariant conditions appeared as a result of the intersection of $\mathcal{M}_{n,n+1,1}$ and $\mathcal{N}_{n,p,l}^d$ in its explicit form.

3. Reductions for the INB lattice

3.1. Restriction of the nth discrete KP hierarchy on $\mathcal{M}_{n,n+1,1}$. INB lattice

Let us consider the invariant submanifold $\mathcal{M}_{n,n+1,1}$ of phase-space $\mathcal{M}$ defined by the algebraic equations

$$J_k^{(n,n+1,1)} = -J_k^{(n+1,n,1)} = a_{k+1}^{[n+1]} - a_{k+1}^{[n]} = 0, \qquad k \geqslant 1 \tag{20}$$

for some fixed positive integer n. Taking into account (6) we can rewrite (20) as

$$\sum_{j=1}^{k-1} a_{k-j}^{[n]}(i) a_j(i+n) + a_k(i+n) = 0. \tag{21}$$

One can easily check that these equations are solved by

$$a_k(i) = a_{k-1}^{[-n]}(i) a_i \tag{22}$$

where $a_i \equiv a_1(i)$. Indeed, substituting the latter in (21) we have

$$a_{i+n} \left(a_{k-1}^{[n]}(i) + \sum_{j=1}^{k-2} a_{k-j-1}^{[n]}(i) a_j^{[-n]}(i+n) + a_{k-1}^{[-n]}(i+n) \right)$$
$$= a_{i+n} a_{k-1}^{[0]}(i) = 0.$$

Here we have used (6). The following technical proposition is valid.

Proposition 1. *In virtue of relations (20)*

$$a_k^{[s]}(i) = \sum_{j=1}^{s} a_{k-1}^{[-n+j-1]}(i) a_{i+j-1}, \qquad \textit{for} \quad s \geqslant 1 \tag{23}$$

and

$$a_k^{[s]}(i) = -\sum_{j=1}^{|s|} a_{k-1}^{[-n-j]}(i) a_{i-j}, \qquad \textit{for} \quad s \leqslant -1.$$

Proof. Taking into account (6) and (22), one has

$$
\begin{aligned}
a_k^{[s+1]}(i) &= a_k^{[s]}(i) + \sum_{j=1}^{k-1} a_{k-j}^{[s]}(i)a_j(i+s) + a_k(i+s) \\
&= a_k^{[s]}(i) + a_{i+s}\left(a_{k-1}^{[s]}(i) + \sum_{j=1}^{k-2} a_{k-j-1}^{[s]}(i)a_j^{[-n]}(i+s) + a_{k-1}^{[-n]}(i+s)\right) \\
&= a_k^{[s]}(i) + a_{k-1}^{[-n+s]}(i)a_{i+s}.
\end{aligned}
$$

Making use of this formula one can successively to prove (23) for $k = 2, 3, \ldots$ by induction with respect to s. □

With (23), we are able to calculate all a_k as discrete functions of a to obtain

$$
a_2(i) = -a_i \sum_{j=1}^{n} a_{i-j}, \qquad a_3(i) = a_i \sum_{j_1=1}^{n} a_{i-j_1}\left(\sum_{j_2=1}^{n+j_1} a_{i-j_2}\right),
$$

$$
a_4(i) = -a_i \sum_{j_1=1}^{n} a_{i-j_1}\left(\sum_{j_2=1}^{n+j_1} a_{i-j_2}\left(\sum_{j_3=1}^{n+j_2} a_{i-j_3}\right)\right)
$$

and so on.

What we learn from the above calculations is that the restriction of the nth discrete KP hierarchy on $\mathcal{M}_{n,n+1,1}$ and subsequent projection $\pi_1 : \mathcal{M} \mapsto \mathcal{M}_1$ generate the hierarchy of evolution equations in the form of differential-difference conservation laws

$$
\partial_s a_i = F_1^{(n,s)}(i+1) - F_1^{(n,s)}(i)
$$

together with conservation laws (11), where conserved densities $\xi_k = \xi_k[a]$ and fluxes $F_k^{(n,s)}[a]$ are some homogeneous polynomials of the kth and $(k+s)$th power, respectively. For the first flow we have

$$
\begin{aligned}
a_i' &= F_1^{(n,1)}(i+1) - F_1^{(n,1)}(i) \\
&= a_2^{[n]}(i+1) - a_2^{[n]}(i).
\end{aligned}
$$

To calculate the right-hand side of this equation as a discrete function of $a = a_i$, it is convenient to use

$$
\begin{aligned}
a_2^{[n+1]}(i) &= a_2^{[n]}(i) + a_1^{[n]}(i)a_{i+n} + a_2(i+n) \\
&= a_2^{[n]}(i+1) + a_1^{[n]}(i+1)a_i + a_2(i),
\end{aligned}
$$

where $a_2(i) = a_1^{[-n]}(i)a_i = -a_1^{[n]}(i-n)a_i$. Taking this into account, we obtain

$$
\begin{aligned}
a_i' &= a_i\left(a_1^{[n]}(i-n) - a_1^{[n]}(i+1)\right) \\
&= a_i\left(\sum_{j=1}^{n} a_{i-j} - \sum_{j=1}^{n} a_{i+j}\right),
\end{aligned}
$$

which is nothing but INB equation (1).

3.2. The functions $S_s^{(n,l)}[a]$ and $T_s^{(n,l)}[a]$ and its properties

Let us prepare, for further use, the discrete functions $S_s^{(n,l)}[a]$ and $T_s^{(n,l)}[a]$ through

$$S_s^{(n,l)}(i) = \sum_{0\leqslant j_{l-1}\leqslant\cdots\leqslant j_0\leqslant s}\left(\prod_{k=0}^{l-1} a_{i+kn+j_k}\right) \tag{24}$$

with $l \geqslant 0$ and $s \geqslant 0$ and

$$T_s^{(n,l)}(i) = \sum_{0\leqslant j_0<\cdots<j_{l-1}\leqslant s}\left(\prod_{k=0}^{l-1} a_{i+kn+j_k}\right)$$

with $l \geqslant 0$ and $s \geqslant l-1$. We observe that the functions $S_s^{(n,l)}$ satisfy the following relations:

$$S_s^{(n,l)}(i) - S_{s-d}^{(n,l)}(i) = \sum_{j=1}^{d} a_{i+s-j+1} S_{s-j+1}^{(n,l-1)}(i+n), \tag{25}$$

$$S_s^{(n,l)}(i) - S_{s-d}^{(n,l)}(i+d) = \sum_{j=1}^{d} a_{i+(l-1)n+j-1} S_{s-j+1}^{(n,l-1)}(i+j-1) \tag{26}$$

for $d = 1, \ldots, s$ and

$$\begin{aligned} S_s^{(n,l)}(i) &= \sum_{j=1}^{s+1} a_{i+s-j+1} S_{s-j+1}^{(n,l-1)}(i+n) \\ &= \sum_{j=1}^{s+1} a_{i+(l-1)n+j-1} S_{s-j+1}^{(n,l-1)}(i+j-1). \end{aligned}$$

For $T_s^{(n,l)}$ we have the identities

$$T_s^{(n,l)}(i) - T_{s-d}^{(n,l)}(i+d) = \sum_{j=1}^{d} a_{i+j-1} T_{s-j}^{(n,l-1)}(i+n+j), \tag{27}$$

$$T_s^{(n,l)}(i) - T_{s-d}^{(n,l)}(i) = \sum_{j=1}^{d} a_{i+(l-1)n+s-j+1} T_{s-j}^{(n,l-1)}(i) \tag{28}$$

with $d = 1, \ldots, s-l+1$ and

$$T_s^{(n,l)}(i) = \sum_{j=1}^{s-l+2} a_{i+j-1} T_{s-j}^{(n,l-1)}(i+n+j) \tag{29}$$

$$= \sum_{j=1}^{s-l+2} a_{i+(l-1)n+s-j+1} T_{s-j}^{(n,l-1)}(i). \tag{30}$$

3.3. A class of reductions for the INB lattice

With $S_s^{(n,l)}$ and $T_s^{(n,l)}$ in hand, we are in position to prove

Proposition 2. *On* $\mathcal{M}_{n,n+1,1}$

$$J_k^{(n,ln+s+1,l)}(i) = T_s^{(n,k+l)}(i-(k-1)n) + \sum_{j=1}^{k-1} a_j^{[-(k-j)n]}(i) T_s^{(n,k+l-j)}(i-(k-j-1)n) \tag{31}$$

for $s \geqslant l$ *and* $J_k^{(n,ln+s+1,l)} \equiv 0$ *for* $s = 0, \ldots, l-1$ *and*

$$J_k^{(n,ln-s-1,l)}(i) = R_s^{(n,k+l)}(i-(k-1)n) + \sum_{j=1}^{k-1} a_j^{[-(k-j)n]}(i) R_s^{(n,k+l-j)}(i-(k-j-1)n) \tag{32}$$

for $s \geqslant 0$*, where* $R_s^{(n,k)}(i) \equiv (-1)^k S_s^{(n,k)}(i-s-1)$.

Let us give some remarks. It is accepted in (31) that $T_s^{(n,k)} \equiv 0$ with $s \leqslant k-2$. For example, $J_k^{(n,ln+l+1,l)} = a_{k-1}^{[-n]} T_l^{(n,l+1)}$. As a corollary of this proposition, one has $\mathcal{M}_{n,n+1,1} \subset \mathcal{M}_{n,ln+s+1,l}$, with $l \geqslant 1$ and $s = 0, \ldots, l-1$. We observe comparing (31) and (32) with (13) that this proposition can be reformulated in the following equivalent form.

Proposition 3. *On* $\mathcal{M}_{n,n+1,1}$

$$q_k^{(n,s+1)}(i) = (-1)^k S_s^{(n,k)}(i-(k-1)n)$$

for $s \geqslant 0$ *and*

$$q_k^{(n,-s-1)}(i) = T_s^{(n,k)}(i-(k-1)n-s-1)$$

for $s \geqslant k-1$ *and* $q_k^{(n,-s-1)} \equiv 0$ *for* $s = 0, \ldots, k-2$.

From this proposition and the relation [12]

$$q_k^{(n,s_1+s_2)}(i) = q_k^{(n,s_1)}(i) + \sum_{j=1}^{k-1} q_j^{(n,s_1)}(i) q_{k-j}^{(n,s_2)}(i+s_1-jn) + q_k^{(n,s_2)}(i+s_1) = (s_1 \leftrightarrow s_2)$$

we obtain two identities

$$S_s^{(n,l)}(i) + \sum_{j=1}^{l-1} (-1)^j S_s^{(n,l-j)}(i) T_s^{(n,j)}(i+(l-j)n) + (-1)^l T_s^{(n,l)}(i) = 0$$

and

$$T_s^{(n,l)}(i) + \sum_{j=1}^{l-1} (-1)^j T_s^{(n,l-j)}(i) S_s^{(n,j)}(i+(l-j)n) + (-1)^l S_s^{(n,l)}(i) = 0$$

establishing the relationship between the discrete functions $S_s^{(n,l)}[a]$ and $T_s^{(n,l)}[a]$.

Proof of proposition 2. To save the space we restrict ourselves by the sketch of the proof. We prove, by induction with respect to k, the validity of (31) for $l = 0$, i.e. that on $\mathcal{M}_{n,n+1,1}$

$$a_k^{[s+1]}(i) = T_s^{(n,k)}(i-(k-1)n) + \sum_{j=1}^{k-1} a_j^{[-(k-j)n]}(i) T_s^{(n,k-j)}(i-(k-j-1)n). \tag{33}$$

In the case $k = 1$ one has $a_1^{[s+1]}(i) = \sum_{j=0}^{s} a_{i+j} = T_s^{(n,1)}(i)$, by definition. Suppose now that the relation (33) is already proved for $k = 1, \ldots, k_0$. Then, for these values of k and arbitrary $m \in \mathbf{Z}$ we can show that

$$a_k^{[m+s+1]}(i) - a_k^{[m]}(i) = T_s^{(n,k)}(i + m - (k-1)n) \\ + \sum_{j=1}^{k-1} a_j^{[m-(k-j)n]}(i) T_s^{(n,k-j)}(i + m - (k-j-1)n).$$

In particular

$$a_k^{[-n+s+1]}(i) - a_k^{[-n]}(i) = T_s^{(n,k)}(i - kn) + \sum_{j=1}^{k-1} a_j^{[-(k-j+1)n]}(i) T_s^{(n,k-j)}(i - (k-j)n). \tag{34}$$

Taking into account (23), we have

$$a_{k+1}^{[s+1]}(i) - a_k^{[-n]}(i) T_s^{(n,1)}(i) = \sum_{j=1}^{s} \left(a_k^{[-n+j]}(i) - a_k^{[-n]}(i)\right) a_{i+j}.$$

Then taking into account (30) and (34) we obtain

$$a_{k+1}^{[s+1]}(i) - a_k^{[-n]}(i) T_s^{(n,1)}(i) = \sum_{j=1}^{k-1} a_{k-j}^{[-(j+1)n]}(i) T_s^{(n,j+1)}(i - jn) + T_s^{(n,k+1)}(i - kn).$$

Therefore we prove that if (33) is valid for $k = 1, \ldots, k_0$, then it is true for $k = k_0 + 1$. Thus, by induction, the relation (31) is proven for $l = 0$. For remaining values $l \geqslant 1$ the functions $J_k^{(n,ln+s+1,l)}$ are calculated with the help of recurrence relation (16). Similar reasonings are used to prove (32). □

With proposition 3 we can easily write equations of the INB lattice hierarchy. To this aim, we use the fact that on $\mathcal{M}_{n,n+1,1}$ one has

$$Q_k^{(n,n+1,1)}(i) = q_{k+1}^{(n,-1)}(i + n + 1) = 0, \qquad \forall k \geqslant 1.$$

Then from (17) we have

$$\partial_s q_1^{(n,-1)}(i) = q_1^{(n,-1)}(i)\left(q_s^{(n,sn)}(i) - q_s^{(n,sn)}(i - n - 1)\right), \tag{35}$$

where $q_1^{(n,-1)}(i) = a_{i-1}$, by definition. According to proposition 3, on $\mathcal{M}_{n,n+1,1}$ one has $q_s^{(n,sn)}(i) = (-1)^s S_{sn-1}^s(i - (s-1)n)$. Substituting the latter in (35) we obtain the evolution equations of the INB hierarchy (2).

Now we would like to write invariant constraints corresponding to the submanifold $\mathcal{M}_{n,n+1,1} \cap \mathcal{N}_{n,p,l}^d$, where d is any divisor of n or $n+1$. From proposition 2, we know that $I_1^{(n,ln+s+1,l)}(i) = T_s^{(n,l+1)}(i)$, for $s \geqslant l$ and $I_1^{(n,ln-s-1,l)}(i) = R_s^{(n,l+1)}(i) = (-1)^{l+1} S_s^{(n,l+1)}(i - s - 1)$ for $s \geqslant 0$. It is natural to require so as intersection $\mathcal{M}_{n,n+1,1} \cap \mathcal{N}_{n,p,l}^d$ to be nontrivial. This requirement means that condition $I_1^{(n,p,l)}(i + d) = I_1^{(n,p,l)}(i)$ must guarantee that $I_k^{(n,p,l)}(i + d) = I_k^{(n,p,l)}(i)$ is identity for all $k \geqslant 2$. Unfortunately, we are able to analyze this only for the case $d = 1$.

Theorem 3. *Each one of the constraints*

$$T_s^{(n,l+1)}(i + 1) = T_s^{(n,l+1)}(i), \qquad s \geqslant l \tag{36}$$

and

$$S_s^{(n,l+1)}(i + 1) = S_s^{(n,l+1)}(i), \qquad s \geqslant 0 \tag{37}$$

is consistent with the INB lattice hierarchy.

From (27) and (28), with $d = 1$, we see that condition (36) can be rewritten as the relation

$$a_{i+ln+s+1}T_{s-1}^{(n,l)}(i+1) = a_i T_{s-1}^{(n,l)}(i+n+1). \tag{38}$$

For (37), taking into account (25) and (26), with $d = 1$, we have the relation

$$a_{i+s+1}S_s^{(n,l)}(i+n+1) = a_{i+ln}S_s^{(n,l)}(i). \tag{39}$$

Proof of theorem 3. To prove theorem, there is a need only to show that intersection $\mathcal{M}_{n,n+1,1} \cap \mathcal{N}_{n,p,l}^1$ is nontrivial. Let us consider the condition (36). We observe that on $\mathcal{M}_{n,n+1,1}$ homogeneous discrete polynomials $I_k^{(n,p,l)}[a]$ are calculated with the help of the recurrence relation

$$I_k^{(n,p,l)}(i) = T_s^{(n,l+k)}(i-(k-1)n) - \sum_{j=1}^{k-1} T_{p+(k-j)n-1}^{(n,j)}(i-(k-1)n)I_{k-j}^{(n,p,l)}(i). \tag{40}$$

Suppose we already proved that in virtue of (36) the equation

$$I_j^{(n,p,l)}(i+1) = I_j^{(n,p,l)}(i) \tag{41}$$

is valid for $j = 1, \ldots, k-1$. Then the relation $I_k^{(n,p,l)}(i+1) = I_k^{(n,p,l)}(i)$ can be written as

$$\begin{aligned} T_s^{(n,l+k)}(i-(k-1)n+1) - T_s^{(n,l+k)}(i-(k-1)n) - \sum_{j=1}^{k-1}\big(T_{p+(k-j)n-1}^{(n,j)}(i-(k-1)n+1) \\ - T_{p+(k-j)n-1}^{(n,j)}(i-(k-1)n)\big)I_{k-j}^{(n,p,l)}(i) = 0. \end{aligned} \tag{42}$$

Making use of the identities (27) and (28), we can rewrite equation (42) as

$$\begin{aligned} a_{i+p}\Bigg(T_{s-1}^{(n,l+k-1)}&(i-(k-1)n+1) - I_{k-1}^{(n,p,l)}(i) \\ &- \sum_{j=1}^{k-2} T_{p+(k-j-1)n-2}^{(n,j)}(i-(k-1)n+1)I_{k-j-1}^{(n,p,l)}(i)\Bigg) \\ &= a_{i-(k-1)n}\Bigg(T_{s-1}^{(n,l+k-1)}(i-(k-2)n+1) - I_{k-1}^{(n,p,l)}(i) \\ &- \sum_{j=1}^{k-2} T_{p+(k-j-1)n-2}^{(n,j)}(i-(k-2)n+1)I_{k-j-1}^{(n,p,l)}(i)\Bigg). \end{aligned}$$

For $k = 1$ the latter coincides with (38). With the help of identities (27), (28) and recurrence relation (40), we can show that the latter is equivalent to the same relation but with k replaced by $k-1$. Therefore, step-by-step, we can show that condition (36) guarantees that (41) is valid for any $j \geqslant 2$. For (37) one can use similar reasonings. □

The following remark is in order. It is easy to prove that the stationarity condition $S_{ln-1}^{(n,l)}(i+n+1) = S_{ln-1}^{(n,l)}(i)$ mentioned in Introduction is equivalent to condition $S_{ln-1}^{(n,l+1)}(i+1) = S_{ln-1}^{(n,l+1)}(i)$ which is the particular case of (37). This example suggests that theorem 3 possibly gives all invariant constraints corresponding to the submanifolds $\mathcal{M}_{n,n+1,1} \cap \mathcal{N}_{n,p,l}^d$.

4. Reductions of the Volterra lattice

4.1. Invariant constraints for the Volterra lattice and its hierarchy

In this section we apply theorem 3 in the important case of the Volterra lattice

$$a_i' = a_i(a_{i-1} - a_{i+1}). \tag{43}$$

Evolution equations of the Volterra lattice hierarchy look as specialization of (2), namely

$$\partial_s a_i = (-1)^s a_i \left(S_{s-1}^s(i-s+2) - S_{s-1}^s(i-s)\right)$$

with

$$S_0^1(i) = a_i, \qquad S_1^2(i) = a_i S_0^1(i+1) + a_{i+1} S_1^1(i+1)$$
$$= a_i a_{i+1} + a_{i+1}(a_{i+1} + a_{i+2}),$$
$$S_2^3(i) = a_i S_0^2(i+1) + a_{i+1} S_1^2(i+1) + a_{i+2} S_2^2(i+1)$$
$$= a_i a_{i+1} a_{i+2} + a_{i+1}\{a_{i+1}a_{i+2} + a_{i+2}(a_{i+2} + a_{i+3})\}$$
$$+\ a_{i+2}\{a_{i+1}a_{i+2} + a_{i+2}(a_{i+2} + a_{i+3}) + a_{i+3}(a_{i+2} + a_{i+3} + a_{i+4})\}$$

and so on.

Let us restrict ourselves in this section by consideration only reductions of the Volterra lattice (43) generated by conditions of the form $T_s^{(1,l+1)}(i+1) = T_s^{(1,l+1)}(i)$. Equation (38) is specified in this case as[3]

$$a_{i+s+l+1} = a_i \frac{T_{s-1}^l(i+2)}{T_{s-1}^l(i+1)}. \tag{44}$$

It should be noted that when $l = 0$, the latter is nothing but the periodicity condition $a_{i+s+1} = a_i$. For some value $i = i_0$, we denote $y_1 = a_i, \ldots, y_{s+l+1} = a_{i+s+l}$—initial data for the discrete equation (44). Then $T_{s-1}^l(i) = T_{s-1}^l(y_1, \ldots, y_{s+l-1})$ is a homogeneous polynomial of the lth power. In what follows we need in

Proposition 4. *The function $T_s^l = T_s^l(y_1, \ldots, y_{s+l})$ is invariant with respect to reversing transformation $y_k \mapsto y_{s+l-k+1}$, i.e.,*

$$T_s^l(y_{s+l}, \ldots, y_1) = T_s^l(y_1, \ldots, y_{s+l}). \tag{45}$$

Proof. By induction with respect to parameter l. For $l = 1$ the relation (45) is evident. Suppose that (45) is proved for some value of l. Then to prove it for $l + 1$ we make use of the identity

$$T_s^{l+1}(y_1, \ldots, y_{s+l+1}) = \sum_{j=1}^{s-l+1} y_j T_{s-j}^l(y_{j+2}, \ldots, y_{s+l+1})$$
$$= \sum_{j=1}^{s-l+1} y_{s+l-j+2} T_{s-j}^l(y_1, \ldots, y_{s+l-j})$$

which stems from (29) and (30). □

Constraining the Volterra lattice (43) by (44) leads to the system of ordinary differential equations

$$y_1' = y_1 \left(y_{s+l+1} \frac{T_{s-1}^l(y_1, \ldots, y_{s+l-1})}{T_{s-1}^l(y_2, \ldots, y_{s+l})} - y_2 \right),$$

[3] Here we use the simplified notation $T_s^{(1,l)} \equiv T_s^l$.

$$y_k' = y_k(y_{k-1} - y_{k+1}), \qquad k = 2, \ldots, s+l, \tag{46}$$
$$y_{s+l+1}' = y_{s+l+1}\left(y_{s+l} - y_1 \frac{T_{s-1}^l(y_3, \ldots, y_{s+l+1})}{T_{s-1}^l(y_2, \ldots, y_{s+l})}\right).$$

Compatibility of (44) with (43) guarantees that the mapping $T : \mathbf{R}^{s+l+1} \mapsto \mathbf{R}^{s+l+1}$ given by

$$T(y_k) = y_{k+1}, \qquad k = 1, \ldots s+l, \qquad T(y_{s+l+1}) = y_1 \frac{T_{s-1}^l(y_3, \ldots, y_{s+l+1})}{T_{s-1}^l(y_2, \ldots, y_{s+l})} \tag{47}$$

yields the discrete symmetry for (46).

Observe that the mapping (47) admits the factorization $T = s_2 \circ s_1$, where s_1 and s_2 are two symmetry transformations acting on variables $\{y_1, \ldots, y_{s+l+1}, x\}$ as

$$s_1(y_k) = y_{s+l-k+1}, \qquad k = 1, \ldots, s+l,$$
$$s_1(y_{s+l+1}) = y_{s+l+1} \frac{T_{s-1}^l(y_1, \ldots y_{s+l-1})}{T_{s-1}^l(y_2, \ldots y_{s+l})}, \qquad s_1(x) = -x$$

and

$$s_2(y_k) = y_{s+l-k+2}, \qquad k = 1, \ldots, s+l+1, \qquad s_2(x) = -x,$$

respectively. With proposition 4, one can easily check that $s_1^2 = s_2^2 = 1$. This is evident, of course, for reversing transformation s_2. This symmetry is the elementary consequence of reversing the symmetry of the Volterra lattice given by the transformation $i \mapsto -i$ and $x \mapsto -x$ and supplemented by appropriate shift $i \mapsto i + \delta$. Having in mind this symmetry one immediately obtains $s_1 = s_2 \circ T$. It is a nontrivial fact only that $s_1^2 = 1$. The question to be posed is: whether the group of discrete symmetry birational transformations generated by s_1 and s_2 covers all birational symmetry transformations for the system (46) or not?

Let us spend some lines to give remarks. It should be noted papers (for example, [1, 2, 11]) where the authors develop a general concept of boundary conditions compatible with higher flows for some integrable lattices. In particular, Adler and Habibullin in [2] showed that Bogoyavlensky–Volterra finite-dimensional systems associated with a simple Lie algebras [4] can be derived as a result of imposing special boundary conditions for the Volterra lattice. Our class of constraints yields finite-dimensional systems (46), including periodic Volterra lattices, which we believe are integrable in Liouville sense.

4.2. *Lax matrices and spectral curves*

Making use of the relation (19) and proposition 3, we derive that in terms of KP wavefunctions, the restriction of the discrete KP hierarchy on $\mathcal{M}_{1,2,1} \cap \mathcal{N}_{1,l+s+1,l}^1$ is defined by a pair of linear equations

$$z\psi_{i+1} - a_i\psi_{i-1} = z\psi_i \qquad \text{and} \qquad z^l\psi_{i+l} + \sum_{j=1}^{l} z^{l-j} T_s^j(i+l-j+1)\psi_{i+l-j} = w\psi_{i+l+s+1}. \tag{48}$$

Here $w = z^l - \sum_{j\geqslant 1} I_j z^{-j}$, where I_j's are values of integrals $I_k^{(1,l+s+1,l)}(i)$. Observe that it plays the role of the Floquet multiplier. It is evident that the pair (48) is equivalent to equation $L\varphi = 0$ on vector-function $\varphi = (\varphi_1, \ldots, \varphi_{l+s+1})$ with some Lax matrix L. Here we denote $\varphi_1 = \psi_i, \ldots, \varphi_{l+s+1} = \psi_{i+l+s}$. One defines the spectral curve by condition $\det L = 0$. It is worth differing two cases: (1) $s = 2g - l$ with $g \geqslant l$ and (2) $s = 2g - l - 1$ with $g \geqslant l + 1$. Calculation shows that in the first case the spectral curve is given by algebraic equation

$$H_0 w^2 + \left(z^{2g+1} + \sum_{j=1}^{g} H_j z^{2g-j+1} \right) w - z^{2g+l+1} - \sum_{j=1}^{l} H_j z^{2g+l-j+1} = 0$$

while in the second case it looks like

$$H_0 w^2 - \left(z^{2g} + \sum_{j=1}^{g} H_j z^{2g-j} \right) w + z^{2g+l} + \sum_{j=1}^{l} H_j z^{2g+l-j} = 0.$$

Rational functions $H_j = H_j(y_1, \ldots, y_{s+l+1})$, by construction, are the first integrals of the system (46).

5. Conclusion

In this paper, by using geometric approach, we have shown a broad class of constraints compatible with dynamics defined by the INB lattice. All these reductions are defined by some conditions which can be represented as the Nth order discrete equation

$$a_{i+N} = R(a_i, \ldots, a_{i+N-1}), \tag{49}$$

with rational right-hand side R. Initial data for the integration INB lattice constrained by (49) is given by vector $\left(y_1^0, \ldots, y_N^0\right) \in \mathbf{R}^N$ which on the one hand gives initial data for discrete equation (49) but on the other hand yields initial data for attached autonomous system of ordinary differential equations such as (46), i.e., $y_k^0 = y_k(x_0)$. In this connection, the first problem to be addressed is the integration of the systems attached to some constraints in such a way as to present in an appropriate form the corresponding solutions of the INB equation. The second problem which we leave for further investigation is to approve these results on other integrable lattices mentioned in a body of the paper.

Acknowledgments

This work has been supported by Russian Foundation for Basic Research grant no. 09-01-00192-a.

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